Anomalous diffusion with absorption: Exact time-dependent solutions

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(Received 5 January 1999; revised manuscript received 30 September 1999)

Recently, analytical solutions of a nonlinear Fokker-Planck equation describing anomalous diffusion with an external linear force were found using a nonextensive thermostatistical Ansatz. We have extended these solutions to the case when an homogeneous absorption process is also present. Some peculiar aspects of the interrelation between the deterministic force, the nonlinear diffusion, and the absorption process are discussed.

I. INTRODUCTION

The ubiquity of the anomalous diffusion phenomenon in nature has attracted the interest of researchers from both the theoretical and experimental point of view. Anomalous diffusion has been found in transport of fluids in porous media and surface growth [1], in NMR relaxometry of liquids in porous glasses [2], and in a two-dimensional fluid flow [3], to name just a few among the large variety of physical phenomena where it is present. A related aspect is the case of density dependent diffusivities as found in some biological systems [4], in polymers [5], and hydrogen diffusion in metals [6] (see also [7]).

Some recent papers have investigated a class of *nonlinear* generalizations of diffusion and Fokker-Planck equations [1,8–10], as a model of correlated anomalous diffusion. Some of those studies were based in a nonextensive thermodynamical formalism [11]. Particularly in Ref. [10] exact solutions for the nonlinear Fokker-Planck equation subject to a linear force have been found. Here we want to show how these solutions can be extended to the case where an absorption process is also present.

We start recalling the ''full'' anomalous diffusion equation or ''nonlinear'' Fokker-Planck equation solved in Ref. [10],

$$\frac{\partial}{\partial t}[p(x,t)]^{\mu} = -\frac{\partial}{\partial x}\{F(x)[p(x,t)]^{\mu}\} + D\frac{\partial^2}{\partial x^2}[p(x,t)]^{\nu}. \tag{1}$$

When $F(x)=0$, Eq. (1) can be interpreted as a diffusion equation for $\Phi(x,t)=[p(x,t)]^{\mu}$, where the diffusivity depends on Φ,

$$\frac{\partial}{\partial t}\Phi = \frac{\partial^2}{\partial x^2}[D(\Phi)\Phi], \tag{2}$$

$$D(\Phi) = \Phi^{(\nu/\mu)-1}. \tag{3}$$

There are several real situations where this power-law dependence of the diffusivity is found. It occurs in the flow of gases through porous media ($\nu/\mu \geqslant 2$ [12]), the flow of water in unsaturated soils ($\nu/\mu = 5$ [13]), the simultaneous diffusion and adsorption in porous samples where the adsorption isotherm is of power-law type [14] ($\nu/\mu \geqslant 1$ for Freundlich type of adsorption isotherm [14,15]). Clearly, in those cases, the diffusivity vanishes (diverges) for $\Phi=0$ when $\nu/\mu>1$ ($\nu/\mu<1$). It is worth remembering that Eq. (1) corresponds to the so called ''porous media equation'' when $\mu=1$ [9,16]. There are a large number of situations where the interest of describing anomalous diffusion plus absorption are of relevance. Notably, it is those related to the diffusion of some (reactive) substance in a gaseous phase through a porous media or a membrane, that react and can be adsorbed in sites inside the pore [17].

Our interest here is to solve the same Eq. (1) but including now terms that describe some kind of absorption process. A general form of such an equation is

$$\frac{\partial}{\partial t}[p(x,t)]^{\mu} = -\frac{\partial}{\partial x}\{F(x)[p(x,t)]^{\mu}\}$$
$$+ D\frac{\partial^2}{\partial x^2}[p(x,t)]^{\nu} - \alpha[p(x,t)]^{\mu'}, \tag{4}$$

where α plays the role of an absorption rate (and becomes the usual one for $\mu'=\mu$). The presence of reaction terms like the one in Eq. (4) (with $\alpha\neq 0$ and $\mu'\neq 0$) is not at all unexpected considering the large amount of work on the problematic of diffusion-limited reactions. Among the diversity of systems that have been studied we only recall here the so-called one species coagulation, that is: $A+A\rightarrow 0$ or $mA \rightarrow lA$ (with $m>l$), that have been associated, among others, with catalytic processes in regular, heterogeneous, or disordered systems [18]. The reaction term may account, in the case $\mu=\mu'$, for an irreversible first-order reaction of the transported substance so that the rate of removal is αC [14]. This extra term also appears when a tracer undergoing radioactive decay is transported through a porous medium, where

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α is the reciprocal of the tracer's mean lifetime [19], as well as in heat flow involving heat production at a rate that is a linear function of the temperature [20]. Finally, in solute transport through adsorbent samples the adsorption rate, at small solute concentration, is usually proportional to the concentration in solution and Eq. (4) applies.

In [10] it has been shown that $p_q(x,t)$, the solution of Eq. (1), for a linear force $F(x)$ has the form

$$p_q(x,t) = \frac{\{1 - \beta(t)(1-q)[x - x_M(t)]^2\}^{1/(1-q)}}{Z_q(t)}, \quad (5)$$

where $q = 1 + \mu - \nu$, $\beta(t)$ depends on the width of the distribution, $x_M(t)$ is the average of the coordinate, and $Z_q(t)$ is a normalization factor. All of them depend on the diffusion parameter D as well as on μ, ν and the force (see [10] for details).

For completeness, as well as for reference, we write Eq. (1) without the ''drift'' $F(x)$ as we will refer to the solutions of such an equation in the following sections:

$$\frac{\partial}{\partial t}[p(x,t)]^\mu = D \frac{\partial^2}{\partial x^2}[p(x,t)]^\nu. \quad (6)$$

According to the results from Ref. [10] we can write that the solution $p_q^{(0)}(x,t)$ of Eq. (6) has the form given in Eq. (5), with

$$x_M(t) = x_0,$$

$$Z_q(t) = \left(\frac{2\nu}{\mu}(\nu + \mu)\pi D t\right)^{1/\nu + \mu},$$

$$\beta(t) = \pi\left(\frac{2\nu}{\mu}(\nu + \mu)\pi D t\right)^{-(2\mu/\nu + \mu)}, \quad (7)$$

where we have used the relation $\beta(0)Z_q(0)^{2\mu} = \pi$, that shall be fulfilled if we want to have a δ-like initial condition.

In the present work we intend to analyze the specific case of a linear drift, namely, $F(x) = k_1 - k_2 x$. This case, where the potential is harmonic (a typical approximation), is the simple nontrivial one where analytic solutions can be obtained just by means of changing the variables to suitable ones, namely, a simple extension of the well known Boltzmann transformation [21].

In Sec. II we start considering the simple case $\mu = \mu'$. We analyze first the case of constant external force ($k_2 = 0$). In this case we can first reduce Eq. (4) by proposing a solution of the form

$$p(x,t) = e^{-(\alpha/\mu)t}\hat{p}(x,t), \quad (8)$$

that yields an equation for $\hat{p}(x,t)$ given by

$$\frac{\partial}{\partial t}[\hat{p}(x,t)]^\mu = -\frac{\partial}{\partial x}\{F(x)[\hat{p}(x,t)]^\mu\}$$

$$+ D e^{\alpha(1-\nu/\mu)t}\frac{\partial^2}{\partial x^2}[\hat{p}(x,t)]^\nu. \quad (9)$$

Although this reduction to a *nonlinear* Fokker-Planck equation like in Eq. (1) looks to always be possible (when k_2

=0), for the case of a linear force (when $k_2 \neq 0$) such a reduction is not possible to make in a simple way and we will need a more general treatment.

The linear force situation, tightly related to the so called Uhlenbeck-Ornstein process ($k_1 = 0; k_2 \neq 0$), is treated in Sec. III, while in Sec. IV we discuss the most general case, that is, when $\mu \neq \mu'$. In the last section we make some final remarks.

II. SOLUTION FOR A CONSTANT FORCE

As indicated above, here we consider the case of a constant force, that is $F(x) = k_1$. Equation (9) can be further reduced making the following change of variables $\xi = x - k_1 t$, that results in

$$\frac{\partial}{\partial t}[\hat{p}(\xi,t)]^\mu = D e^{\alpha(1-\nu/\mu)t}\frac{\partial^2}{\partial \xi^2}[\hat{p}(\xi,t)]^\nu. \quad (10)$$

Now we change the time variable according to

$$\hat{p}(\xi,t) \leftrightarrow \hat{p}(\xi,z(t)) \Rightarrow \frac{\partial}{\partial t} = \dot{z}(t)\frac{\partial}{\partial z}, \quad (11)$$

$$z(t) = \int_0^t e^{\alpha(1-\nu/\mu)\tau}d\tau = \frac{1 - e^{-\gamma t}}{\gamma}, \quad t \geq 0, \quad (12)$$

with $\gamma = -\alpha(\mu - \nu)/\mu$, and obtain the following equation valid for all $t \geq 0$

$$\frac{\partial}{\partial z}[\hat{p}(\xi,z)]^\mu = D \frac{\partial^2}{\partial \xi^2}[\hat{p}(\xi,z)]^\nu. \quad (13)$$

Hence, if $p_q^{(0)}(x,t)$ is the solution of Eq. (6), the solution with $F(x) = k_1$ plus absorption results to be

$$p_q(x,t) = e^{-\frac{\alpha}{\mu}t}p_q^{(0)}(x - k_1 t, z(t)). \quad (14)$$

It is easy to check that this solution has the right limits for $\alpha \to 0$ and for $\alpha > 0$ and $\mu = \nu = 1$, i.e., the standard Fokker-Planck equation (plus absorption).

The new variable $z(t)$ plays the role of an *effective time* for the dispersion process. It exhibits markedly different behaviors depending on the ratio μ/ν. The case $\mu/\nu > 1$ ($\gamma < 0$) corresponds to *superdiffusive* transport when there is no absorption [10]. When absorption is present this *superdiffusion* is enhanced since the effective time z grows exponentially as a function of the real time t.

In the case $\mu/\nu < 1$ ($\gamma > 0$), which leads to *subdiffusion* for $\alpha = 0$, the presence of absorption also plays a key role. The effective time $z(t)$ converges to an asymptotic value: $\lim_{t\to\infty}z(t) = z_\infty = 1/\gamma$. Therefore, the distribution $p_q^{(0)}(\xi,z)$ evolves toward an asymptotic curve $p_q^{(0)}(\xi,z_\infty)$. Let us note that z_∞ diverges whenever $\alpha \to 0$ or $\mu/\nu \to 1$. In Fig. 1 we compare, in the $\mu/\nu < 1$ case, the time evolution of the distributions for $\alpha \neq 0$ and $\alpha = 0$. We also compare these distributions with the shape of the $\alpha \neq 0$ asymptotic curve. For completeness, we include in Fig. 2 the behavior of z on t, also illustrating its dependence on μ/ν. Finally in the case
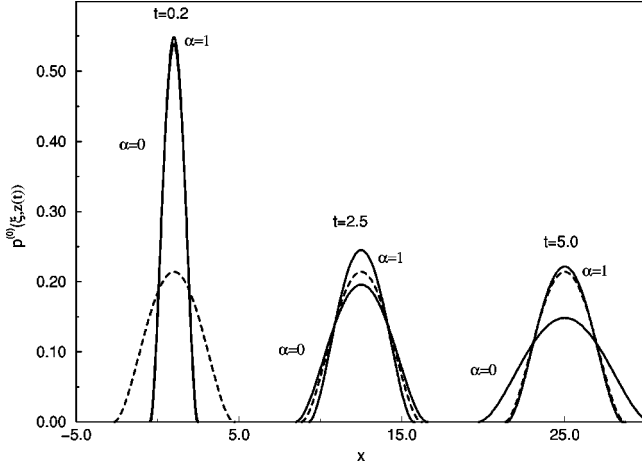
FIG. 1. Evolution of the distribution $p_q^{(0)}(\xi,z(t))$, as given in Eq. (14), in the case $\mu/\nu=2/3<1$. It is shown at times $t=0.2$, 2.5, and 5 for $\alpha=0$ and $\alpha=1$ ($\mu=1,\nu=1.5,D=1,k_1=5,q=0.5$). The dashed line corresponds to the $\alpha=1$ asymptotic distribution shape.

$\mu=\nu$ (*normal diffusion+absorption* for the quantity $\Phi(x,t)=[p(x,t)]^\mu$) the change of variables becomes a simple time scaling.

As already mentioned, Eq. (4) can be viewed as a classical diffusion equation for $\Phi(x,t)=[p(x,t)]^\mu$, where the diffusivity depends on $\Phi(x,t)$ through $D(\Phi)=D\Phi^{\nu/\mu-1}$. Therefore, it becomes clear that the absorption can enhance (reduce) the diffusive transport whenever $\mu/\nu>1$ ($\mu/\nu<1$), namely, as absorption proceeds Φ decreases, yielding to an increase or not in $D(\Phi)$ according to the μ/ν ratio. This qualitative description seems to be in complete agreement with the previous quantitative results.

III. SOLUTION FOR A LINEAR FORCE CASE

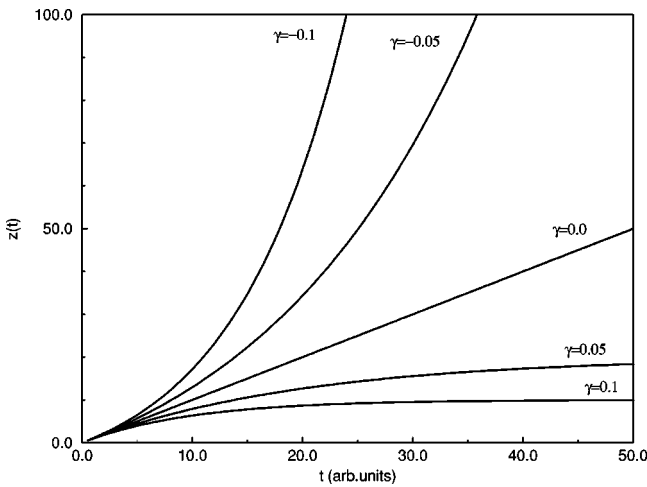We now consider the case of a linear force, given by $F(x)=k_1-k_2x$ (here and in what follows we assume that

FIG. 2. Time dependence of the *effective time* z for several values of γ. It is apparent that $z(t)$ saturates at a finite value as $t\rightarrow\infty(z_\infty=1/\gamma)$ whenever $\gamma>0$. In the case $\gamma=0$ the effective time becomes equal to the real time: $z=t$. For $\gamma<0$ the effective time grows exponentially with t.

$k_2>0$), whose general solution without absorption was found in [10]. To start with, we assume that $\mu=\mu'$. With the hint of the change of variables made in the previous section we propose the following changes, that define the new variables

$$p(x,t)=e^{-wt}\hat{p}(\xi,z(t)), \tag{15}$$

$$\xi=xg(t)+f(t), \tag{16}$$

with $w,g(t)$ and $f(t)$ to be determined. In terms of these new variables the time and space derivatives becomes

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial t}+(x\dot{g}(t)+\dot{f}(t))\frac{\partial}{\partial\xi}+\dot{z}(t)\frac{\partial}{\partial z}, \tag{17}$$

$$\frac{\partial}{\partial x}=g(t)\frac{\partial}{\partial\xi}. \tag{18}$$

Taking into account these results and the form proposed for the solution [given by Eq. (15)], each separate term of Eq. (4) becomes

$$\frac{\partial}{\partial t}[p(x,t)]^\mu=-w\mu e^{-w\mu t}[\hat{p}(\xi,z)]^\mu+e^{-w\mu t}(x\dot{g}(t)$$
$$+\dot{f}(t))\frac{\partial}{\partial\xi}[\hat{p}(\xi,z)]^\mu$$
$$+e^{-w\mu t}\dot{z}(t)\frac{\partial}{\partial z}[\hat{p}(\xi,z)]^\mu, \tag{19}$$

$$-\frac{\partial}{\partial x}\{(k_1-k_2x)[p(x,t)]^\mu\}$$
$$=k_2[p(x,t)]^\mu-(k_1-k_2x)\frac{\partial}{\partial x}[p(x,t)]^\mu$$
$$=e^{-w\mu t}\left\{k_2[\hat{p}(\xi,z)]^\mu-(k_1-k_2x)g(t)\frac{\partial}{\partial\xi}[\hat{p}(\xi,z)]^\mu\right\}, \tag{20}$$

$$D\frac{\partial^2}{\partial x^2}[p(x,t)]^\nu=Dg^2(t)e^{-w\nu t}\frac{\partial^2}{\partial\xi^2}[\hat{p}(\xi,z)]^\nu. \tag{21}$$

In this way, the equation we obtain for $\hat{p}(\xi,z)$ replacing into Eq. (4) with $F(x)=k_1-k_2x$ results (after arranging terms and multiplying by $e^{w\mu t}$)

$$\dot{z}(t)\frac{\partial}{\partial z}[\hat{p}(\xi,z)]^\mu=[w\mu+k_2-\alpha][\hat{p}(\xi,z)]^\mu$$
$$-[(k_1-k_2x)g(t)+x\dot{g}(t)+\dot{f}(t)]\frac{\partial}{\partial\xi}$$
$$\times[\hat{p}(\xi,z)]^\mu+Dg^2(t)e^{w(\mu-\nu)t}\frac{\partial^2}{\partial\xi^2}$$
$$\times[\hat{p}(\xi,z)]^\nu. \tag{22}$$

In order to reduce the last equation to one with a form similar to Eq. (6), we need to cancel the first two terms on the rhs, and reduce the coefficient of the third one to a constant. To operate with the second term, we shall cancel it for all values of x. These conditions yield the following equations:

$$0 = w\mu + k_2 - \alpha, \tag{23}$$

$$0 = -k_2 g(t) + \dot{g}(t), \tag{24}$$

$$0 = k_1 g(t) + \dot{f}(t), \tag{25}$$

$$1 = g^2(t) e^{w(\mu-\nu)t} \dot{z}(t)^{-1}, \tag{26}$$

rendering

$$w = (-k_2 + \alpha)\mu^{-1}, \tag{27}$$

$$g(t) = G e^{k_2 t}, \tag{28}$$

$$f(t) = H - G \frac{k_1}{k_2} e^{k_2 t}, \tag{29}$$

$$z(t) - z(0) = G^2 \{1 - e^{-\gamma t}\} \gamma^{-1}, \tag{30}$$

with $\gamma = [-k_2(\mu+\nu) + \alpha(\nu-\mu)]\mu^{-1}$. In the general case, the values of the constants shall be chosen to fulfill some particular initial condition. Here, to simplify, we choose $G = 1$, implying that we do not change the x scale at $t=0$. Also, in order to make the change of space variables in such a way to have it centered at the potential minimum $[\xi' = (x - k_1/k_2)]$ we adopt $H=0$. Finally we choose $z(0)=0$ to preserve the time origin. With these values we have

$$\xi = \left(x - \frac{k_1}{k_2}\right) e^{k_2 t}, \tag{31}$$

$$z(t) = \{1 - e^{-\gamma t}\} \gamma^{-1}, \tag{32}$$

and the solution of Eq. (4) with $F(x) = k_1 - k_2 x$ is

$$p_q(x,t) = e^{[(k_2-\alpha)/\mu]t} p_q^{(0)}(\xi,z). \tag{33}$$

This is the final result for the present case ($\mu = \mu'$). It is trivial to check its validity in some limits, the most obvious one is to choose $\alpha=0$, recovering the solution of Ref. [10]. With $\mu = \nu = 1$ and $\alpha \neq 0$ we recover the simple case of diffusion in a harmonic potential with absorption. Also, if we consider the case $\mu = \nu$ and $\alpha \neq 0$, it is immediate to obtain (remember that $\mu = \nu$ gives $q=1$!)

$$p_1(x,t) = e^{-\alpha/\mu t} \frac{e^{-k_2/2\mu D[x-(k_1/k_2)-x_0 e^{-k_2 t}]^2/(1-e^{-2k_2 t})}}{\left[\frac{2\pi\mu D}{k_2}(1-e^{-k_2 t})\right]^{1/2\mu}}. \tag{34}$$

This result becomes obvious after making the change $p_1(x,t)^\mu = \phi(x,t)$, reducing the problem to an effective one of diffusion in a harmonic potential with absorption for $\phi(x,t)$. Clearly, even though the solution has a Gaussian form (times a decaying exponential term), the width of the
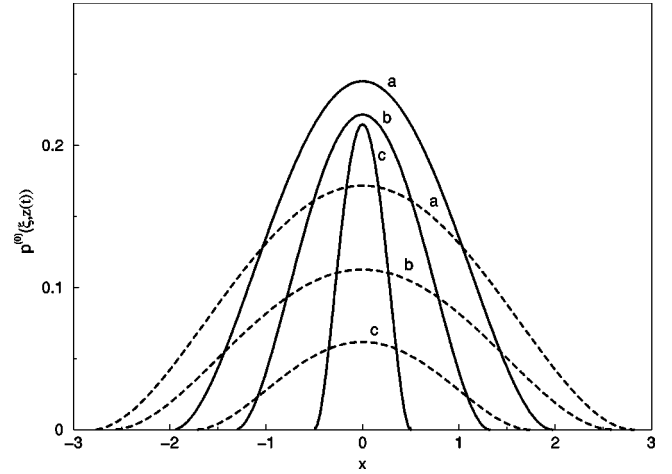

FIG. 3. Evolution of the distribution $p_q^{(0)}[\xi,z(t)]$, as given in Eq. (33), in the case $\mu/\nu = 2 > 1$. It is shown at times $t = 0.5$, 0.8, and 1.0 for $\alpha = 0$ and $\alpha = 5$ ($\mu = 1, \nu = 0.5, D = 1, k_1 = 0, k_2 = 1.0, q = 1.5$). The dotted lines correspond to the $\alpha = 0$ distribution, while the solid lines correspond to the $\alpha = 5$ case.

Gaussian factor behaves "anomalously" as it differs from the one in the associated Ornstein-Uhlenbeck process [22].

As in the constant-force case, absorption process markedly influences the time evolution of dispersion. A straightforward calculation yields the dispersion of the distribution in the present case (i.e., $\mu = \mu'$)

$$\langle (x - \langle x \rangle)^2 \rangle = \frac{1}{\beta[z(t)]} e^{-2k_2 t}. \tag{35}$$

In the superdiffusive case ($\mu/\nu > 1$) $\beta[z(t)]$ becomes asymptotically exponential, namely, $\beta[z(t)] \propto \exp(-2\{[k_2(\mu+\nu) + \alpha(\mu-\nu)]/\mu+\nu\}t)$. Replacing this result in Eq. (35) we obtain the long time behavior of the dispersion

$$\langle (x - \langle x \rangle)^2 \rangle \propto e^{2\alpha[(\mu-\nu)/(\mu+\nu)]t}. \tag{36}$$

Therefore, the *superdiffusive* transport enhanced by absorption yields an exponentially increasing dispersion even in an attractive potential. The *subdiffusive* case ($\mu/\nu < 1$) presents two different situations. Although in both cases the dispersion decays exponentially, when the absorption rate is small $[\gamma < 0, \alpha < k_2(\nu+\mu)/(\nu-\mu)]$, absorption is the rate controlling process for dispersion

$$\langle (x - \langle x \rangle)^2 \rangle \propto e^{-2\alpha[(\nu-\mu)/(\nu+\mu)]t}. \tag{37}$$

In the other case, when the absorption rate is large ($\gamma > 0$), the attractive force becomes rate limiting for dispersion process

$$\langle (x - \langle x \rangle)^2 \rangle \propto e^{-2k_2 t}. \tag{38}$$

In order to compare the influence of the absorption term on the solutions we have found, in Fig. 3 we depict the solution given in Eq. (33), in the case $\mu/\nu > 1$ (superdiffusion), for $\alpha = 0$ and $\alpha \neq 0$ at different times. In Fig. 4 we compare, in the subdiffusive case, the solution in Eq. (33) when absorption is the rate limiting process for dispersion to
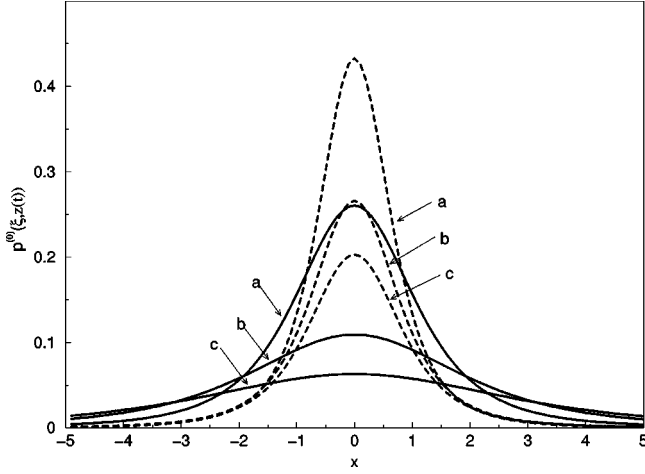
FIG. 4. Evolution of $p_q^{(0)}[\xi, z(t)]$, as given by Eq. (33), in the case $\mu/\nu = 2/3 < 1$. We show it at times $t = 0.2$, 1, and 5 for $\alpha = 0.5$ and $\alpha = 2.0$ ($\mu = 1, \nu = 1.5, D = 1, k_1 = 0, k_2 = 0.2, q = 0.5$). The dashed lines correspond to the case $\alpha = 0.5$, where the attractive force becomes dispersion rate limiting. The solid lines correspond to $\alpha = 2.0$, where absorption is the rate controlling process for dispersion.

the case when the attractive force controls dispersion. In both figures the differences between the characteristics of the different situations are apparent.

IV. GENERAL ABSORPTION TERM

In this section we consider Eq. (4), in the general case $\mu \neq \mu'$. The following "simple" kinetic equation

$$\frac{\partial}{\partial t}[p(t)]^\mu = -\alpha[p(t)]^{\mu'}, \qquad (39)$$

whose solution is

$$p(t) = \left(1 - (1-q')\frac{\alpha}{\mu}t\right)^{-[1/(1-q')]}, \qquad (40)$$

where $q' = 1 - \mu' + \mu$ strongly indicates the replacement of the exponential in the change of variables in Eqs. (14) and (15), by the q'-exponential function defined by Eq. (40) [23]. The ordinary exponential function is recovered when $q' \rightarrow 1$. If we try this possibility, together with the Ansatz in Eq. (7), it immediately leads to the condition $\mu = \mu'$. This result becomes apparent when analyzing Eq. (1) in terms of $\Phi(x,t) = [p(x,t)]^\mu$. The form proposed in Eq. (8) allows us to reduce the general equation, eliminating the absorption term, only when absorption is proportional to $\Phi(x,t)$. However, the general case with $\mu \neq \mu'$ will have a solution whose scaling properties can be determined.

To find such scaling behavior it is enough to consider the simplified situation without external force, that is

$$\frac{\partial}{\partial t}[p(x,t)]^\mu = D\frac{\partial^2}{\partial x^2}[p(x,t)]^\nu - \alpha[p(x,t)]^{\mu'}. \qquad (41)$$

We consider the following Ansatz

$$p(x,t) = \varphi(t)\Theta(\xi), \qquad (42)$$

with $\xi = \psi(t)x$. Replacing this into Eq. (41), we obtain the functions $\varphi(t)$ and $\psi(t)$ as

$$\varphi(t) = [1 + (1-q')(\alpha/\mu)t]^{-[1/(1-q')]}, \qquad (43)$$

$$\psi(t) = [1 + (1-q')(\alpha/\mu)t]^{-1/2[\mu'-\nu)/(1-q')]}, \qquad (44)$$

where $q' = 1 - \mu' + \mu$. Hence, Eq. (41) for $\Theta(\xi)$ reduces to an ordinary differential equation on the variable ξ

$$\Theta^\mu + \left[\frac{\mu'-\nu}{2\mu}\right]\xi\frac{d}{d\xi}\Theta^\mu = D\frac{d^2}{d\xi^2}\Theta^\nu + \Theta^{\mu'}. \qquad (45)$$

Once again, the previous results can be interpreted in terms of $\Phi(x,t)$. Writing the absorption term as $-(\alpha\Phi^{\mu'/\mu-1})\Phi$, it can be seen that in the case $\mu' < \mu$ the absorption process is enhanced as Φ decreases with time. This leads to a finite time $t_c = \mu/\alpha(\mu - \mu')$, where Φ becomes zero. On the other hand, when $\mu' > \mu$ we can obtain the asymptotic dispersion, even though the ordinary differential equation for $\Theta(\xi)$, Eq. (45), is too complicated to be solved analytically. However, for the nth moment of the distribution, we obtain

$$\langle x^{2n}\rangle = \left[\int dx x^{2n}p(x,t)\right]\bigg/\left[\int dx p(x,t)\right]$$

$$= \left[\int dx x^{2n}\varphi(t)\Theta(\psi(t)x)\right]\bigg/\left[\int dx\varphi(t)\Theta(\psi(t)x)\right]$$

$$= \psi(t)^{-2n}\left[\int d\xi \xi^{2n}\Theta(\xi)\right]\bigg/\left[\int d\xi\Theta(\xi)\right]$$

$$= \psi(t)^{-2n}A_{2n}, \qquad (46)$$

$$\langle x^{2n+1}\rangle = 0, \qquad (47)$$

yielding

$$\langle (x-\langle x\rangle)^2\rangle = \langle x^2\rangle = \psi(t)^{-2}A_2 \sim t^{[(\mu'-\nu)/(1-q')]}$$

$$= t^{(\mu'-\nu/\mu'-\mu)}. \qquad (48)$$

Hence, it is clear that, as in the previous case ($\mu' = \mu$), $\mu/\nu < 1$ corresponds to subdiffusion, whereas the case $\mu/\nu > 1$ corresponds to superdiffusive transport.

V. FINAL REMARKS

The Fokker-Planck equation that was generalized to a nonextensive scenario [8] has been further generalized to include the possibility of an absorption process. We have shown that the exact solutions of Eq. (4) (a nonlinear Fokker-Planck equation subject to linear forces) found in [10] when $\alpha = 0$, can be extended for the case $\alpha \neq 0$ and $\mu' = \mu$. However, in the general case $\mu' \neq \mu$, we have been only able to obtain the scaling properties of the solution (whose analytical form we cannot obtain), and the asymptotic behavior of the whole hierarchy of moments.

Summarizing our results for the nonlinear process of anomalous diffusion plus absorption, as described by Eq. (4), we have found that the solution (i) in a constant force field

and for $\mu/\nu > 1$ shows a superdiffusive behavior that is *enhanced* when $\alpha \neq 0$ ($\gamma < 0$); (ii) when $\gamma > 0$ ($\mu/\nu < 1$) the concentration reaches an asymptotic constant profile; (iii) for a linear force and $\gamma < 0$, we find an exponentially increasing dispersion for superdiffusion; (iv) in the linear force case, an exponentially decreasing dispersion arises for subdiffusion ($\gamma > 0$), where absorption is the rate controlling process for dispersion when absorption rate is small [$\alpha < k_2(\nu + \mu)/(\nu - \mu)$], while the attractive force becomes the rate limiting dispersion process when absorption is large enough [$\alpha > k_2(\nu + \mu)/(\nu - \mu)$].

The present results give further support to the argument [10] that a generalized thermostatistics including nonextensivity constitutes an adequate framework within which it is possible to unify both normal and correlated anomalous diffusion [10], extended now to the case when an absorption process is also present. Also, as indicated in [9], this kind of work points out the convenience of paying more attention to the thermodynamic aspects of non-Fickian diffusion. More-over, it has been suggested [10] that even Levy-like anomalous diffusion [24] (that can be discussed by means of linear Fokker-Planck equations with fractional derivatives) can be included within the present common framework of nonlinear Fokker-Planck equations with fractional derivatives.

This work also opens the possibility of analyzing reaction-diffusion systems on a fractal substratum, by considering the nonlinear Fokker-Planck equation with other forms of reaction terms. This problem will be the subject of further work.

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